Harmonic Balance with Small Signal Perturbation

Kai Bittner, Hans Georg Brachtendorf and Martin Steiger

University of Applied Sciences of Upper Austria
Hardware Software Design
FH-OÖ/Hagenberg

29. Juli 2021

Periodic Steady States (PSS)

- Solve circuit equations

$$\frac{d}{dt}q(x(t)) + \underbrace{i(x(t)) + s(t)}_{f(x(t),t)} = 0$$

$x(t), s(t), q(x), i(x) \in \mathbb{R}^n$

- with periodic input

$$s(t) = s(t+T),$$

e.g. $s(t) = c\ \sin\left(\frac{2\pi t}{T}\right)$

- under periodicity constrain

$$x(t) = x(t+T)$$

Harmonic Balance (HB)

- Approximate solution by truncated Fourier series

$$x(t) = \sum_{k=-N}^{N} X_k \, e^{j\omega t}, \qquad \omega = \frac{2\pi}{T}$$

Harmonic Balance (HB)

- Approximate solution by truncated Fourier series

$$x(t) = \sum_{k=-N}^{N} X_k \, e^{j\omega t}, \qquad \omega = \frac{2\pi}{T}$$

- Galerkin discretization

$$j\omega\ell \int_0^T q\big(x(t)\big)\, e^{-j\omega\ell t}\, dt + \int_0^T f\big(x(t), t\big)\, e^{-j\omega\ell t}\, dt = 0, \quad \ell = -N, \ldots, N$$

Harmonic Balance (HB)

- Approximate solution by truncated Fourier series

$$x(t) = \sum_{k=-N}^{N} X_k \, e^{j\omega t}, \qquad \omega = \frac{2\pi}{T}$$

- Galerkin discretization

$$j\omega\ell \int_0^T q(x(t))\, e^{-j\omega\ell t}\, dt + \int_0^T f(x(t),t)\, e^{-j\omega\ell t}\, dt = 0, \quad \ell = -N, \ldots, N$$

- Numerical integration (Trapezoidal Rule) results in nonlinear system

$$\Omega\, \mathcal{F} q(\mathcal{F}^H \boldsymbol{X}) + \mathcal{F} f(\mathcal{F}^H \boldsymbol{X}) = \Omega\, Q(\boldsymbol{X}) + F(\boldsymbol{X}) = 0$$

Harmonic Balance (HB)

- Approximate solution by truncated Fourier series

$$x(t) = \sum_{k=-N}^{N} X_k \, e^{j\omega t}, \qquad \omega = \frac{2\pi}{T}$$

- Galerkin discretization

$$j\omega\ell \int_0^T q(x(t))\, e^{-j\omega\ell t}\, dt + \int_0^T f(x(t),t)\, e^{-j\omega\ell t}\, dt = 0, \quad \ell = -N, \ldots, N$$

- Numerical integration (Trapezoidal Rule) results in nonlinear system

$$\Omega\, \mathcal{F} q(\mathcal{F}^H \boldsymbol{X}) + \mathcal{F} f(\mathcal{F}^H \boldsymbol{X}) = \Omega\, Q(\boldsymbol{X}) + F(\boldsymbol{X}) = 0$$

- Solved by Newton's method

2-Tone Signals

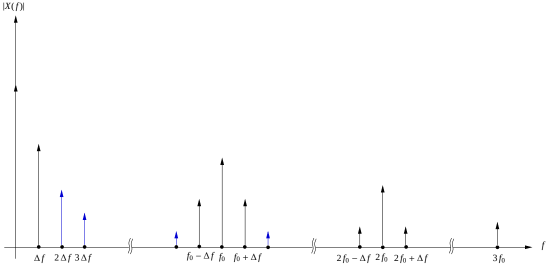
- $x(t) := \hat{x}(t,t), \quad \hat{x}(t_1,t_2) = \hat{x}(t_1+T_1,t_2) = \hat{x}(t_1,t_2+T_2)$

2-Tone Signals

- $x(t) := \hat{x}(t,t), \qquad \hat{x}(t_1,t_2) = \hat{x}(t_1+T_1,t_2) = \hat{x}(t_1,t_2+T_2)$
- $$x(t) = \sum_{k,\ell\in\mathbb{Z}} X_{k,\ell} e^{j(k\omega_1+\ell\omega_2)t}$$

2-Tone Signals

- $x(t) := \hat{x}(t,t), \qquad \hat{x}(t_1,t_2) = \hat{x}(t_1+T_1,t_2) = \hat{x}(t_1,t_2+T_2)$
- $$x(t) = \sum_{k,\ell\in\mathbb{Z}} X_{k,\ell} e^{j(k\omega_1+\ell\omega_2)t}$$
- Sparse spectrum



$\Delta f = f_2 - k f 1, \ k \in \mathbb{Z}$

Multi-rate PDE

Multi tone source term $s(t) = \hat{s}(t, t)$

Multi-rate PDE

Multi tone source term $s(t) = \hat{s}(t, t)$

Solve

$$\frac{\partial}{\partial t_1} q\big(\hat{x}(t_1, t_2)\big) + \frac{\partial}{\partial t_2} q\big(\hat{x}(t_1, t_2)\big) + i\big(\hat{x}(t_1, t_2)\big) + \hat{s}(t_1, t_2) = 0$$

with $\hat{x}(t_1, t_2) = \hat{x}(t_1 + T_1, t_2) = \hat{x}(t_1, t_2 + T_2)$

Multi-rate PDE

Multi tone source term $s(t) = \hat{s}(t, t)$

Solve

$$\frac{\partial}{\partial t_1} q\big(\hat{x}(t_1, t_2)\big) + \frac{\partial}{\partial t_2} q\big(\hat{x}(t_1, t_2)\big) + i\big(\hat{x}(t_1, t_2)\big) + \hat{s}(t_1, t_2) = 0$$

with $\hat{x}(t_1, t_2) = \hat{x}(t_1 + T_1, t_2) = \hat{x}(t_1, t_2 + T_2)$

$x(t) = \hat{x}(t, t)$ is the quasi-periodic steady state solution.

2-tone HB

- Approximate solution by truncated Fourier series

$$\hat{x}(t_1, t_2) = \sum_{\|(k,\ell)\| \le N} X_{k,\ell} e^{j(k\omega_1 t_1 + \ell\omega_2 t_2)}$$

2-tone HB

- Approximate solution by truncated Fourier series

$$\hat{x}(t_1, t_2) = \sum_{\|(k,\ell)\| \leq N} X_{k,\ell} e^{j(k\omega_1 t_1 + \ell\omega_2 t_2)}$$

- As before discretization leads to a nonlinear system

$$j(\omega_1 k + \omega_2 \ell)\, Q_{k,\ell}(X) + F_{k,\ell}(X) = 0, \qquad \|(k,\ell)\| \leq N$$

Small signal perturbation technique

- $x_0(t)$ PSS for source term $s_0(t)$

Small signal perturbation technique

- $x_0(t)$ PSS for source term $s_0(t)$
- Small perturbation $s(t) = s_0(t) + \Delta s(t)$ of source leads to perturbation $x(t) = x_0(t) + \Delta x(t)$ of solution.

Small signal perturbation technique

- $x_0(t)$ PSS for source term $s_0(t)$
- Small perturbation $s(t) = s_0(t) + \Delta s(t)$ of source leads to perturbation $x(t) = x_0(t) + \Delta x(t)$ of solution.
- Linearization yields

$$\frac{d}{dt}\Big(\underbrace{\tfrac{dq}{dx}\big(x_0(t)\big)}_{C(t)}\Delta x(t)\Big) + \underbrace{\tfrac{di}{dx}\big(x_0(t)\big)}_{G(t)}\Delta x(t) + \Delta s(t) = 0$$

Small signal perturbation technique

- $x_0(t)$ PSS for source term $s_0(t)$
- Small perturbation $s(t) = s_0(t) + \Delta s(t)$ of source leads to perturbation $x(t) = x_0(t) + \Delta x(t)$ of solution.
- Linearization yields

$$\frac{d}{dt}\Big(\underbrace{\tfrac{dq}{dx}(x_0(t))}_{C(t)}\Delta x(t)\Big) + \underbrace{\tfrac{di}{dx}(x_0(t))}_{G(t)}\Delta x(t) + \Delta s(t) = 0$$

- Linear non-stationary ODE

Small signal perturbation technique

- $\Delta s = \hat{s}\, e^{j(n\omega + \Delta\omega)t}$

Small signal perturbation technique

- $\Delta s = \hat{s}\, e^{j(n\omega + \Delta\omega)t}$
- Fourier Expansion:
  $C(t) = \sum_{k\in\mathbb{Z}} C_k e^{jk\omega t}$, $G(t) = \sum_{k\in\mathbb{Z}} G_k e^{jk\omega t}$, $x(t) = \sum_{k\in\mathbb{Z}} X_k e^{j(k\omega + \Delta\omega)t}$

Small signal perturbation technique

- $\Delta s = \hat{s}\, e^{j(n\omega + \Delta\omega)t}$
- Fourier Expansion:
  $C(t) = \sum_{k\in\mathbb{Z}} C_k e^{jk\omega t}$, $G(t) = \sum_{k\in\mathbb{Z}} G_k e^{jk\omega t}$, $x(t) = \sum_{k\in\mathbb{Z}} X_k e^{j(k\omega+\Delta\omega)t}$

$$j(\omega\ell + \Delta\omega) \sum_{k\in\mathbb{Z}} C_{\ell-k} X_k + \sum_{k\in\mathbb{Z}} G_{\ell-k} X_k = \hat{s}\delta_{kn}$$

Small signal perturbation technique

- $\Delta s = \hat{s}\, e^{j(n\omega+\Delta\omega)t}$
- Fourier Expansion:
  $C(t) = \sum_{k\in\mathbb{Z}} C_k e^{jk\omega t}$, $G(t) = \sum_{k\in\mathbb{Z}} G_k e^{jk\omega t}$, $x(t) = \sum_{k\in\mathbb{Z}} X_k e^{j(k\omega+\Delta\omega)t}$

$$j(\omega\ell+\Delta\omega)\sum_{k\in\mathbb{Z}} C_{\ell-k} X_k + \sum_{k\in\mathbb{Z}} G_{\ell-k} X_k = \hat{s}\delta_{kn}$$

- Truncation and Approximation of C_k and G_k yields a finite linear system

Small signal perturbation technique

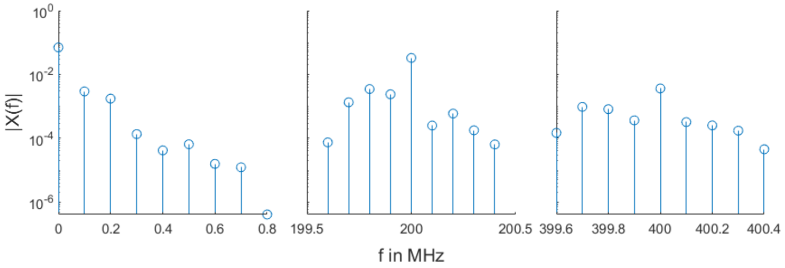
- $\Delta s = \hat{s}\, e^{j(n\omega+\Delta\omega)t}$
- Fourier Expansion:
  $C(t) = \sum_{k\in\mathbb{Z}} C_k e^{jk\omega t},\ G(t) = \sum_{k\in\mathbb{Z}} G_k e^{jk\omega t},\ x(t) = \sum_{k\in\mathbb{Z}} X_k e^{j(k\omega+\Delta\omega)t}$

$$j(\omega\ell + \Delta\omega) \sum_{k\in\mathbb{Z}} C_{\ell-k} X_k + \sum_{k\in\mathbb{Z}} G_{\ell-k} X_k = \hat{s}\delta_{kn}$$

- Truncation and Approximation of C_k and G_k yields a finite linear system
- X-Parameter computation for $\Delta\omega = 0$ possible.

Test with Gilbert mixer

- LO Input: 100MHz
- RF Input: 99.9MHz $\rightsquigarrow$ $\Delta f = 0.1MHz$
- 2-tone Output:



HB vs. Perturbation

Sweep over RF amplitude