Memoryless non-linear L2-Systems II
Tanh Non-Linearity

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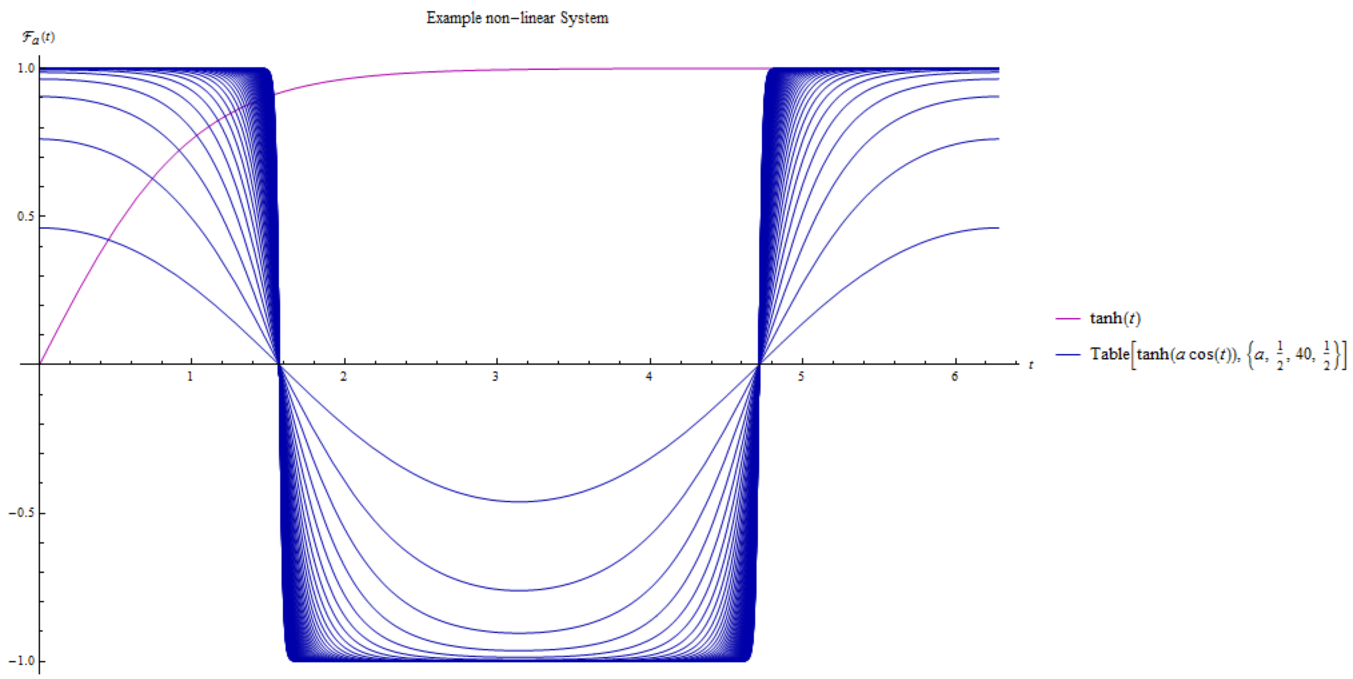
Tanh Non-Linearity (Domain)



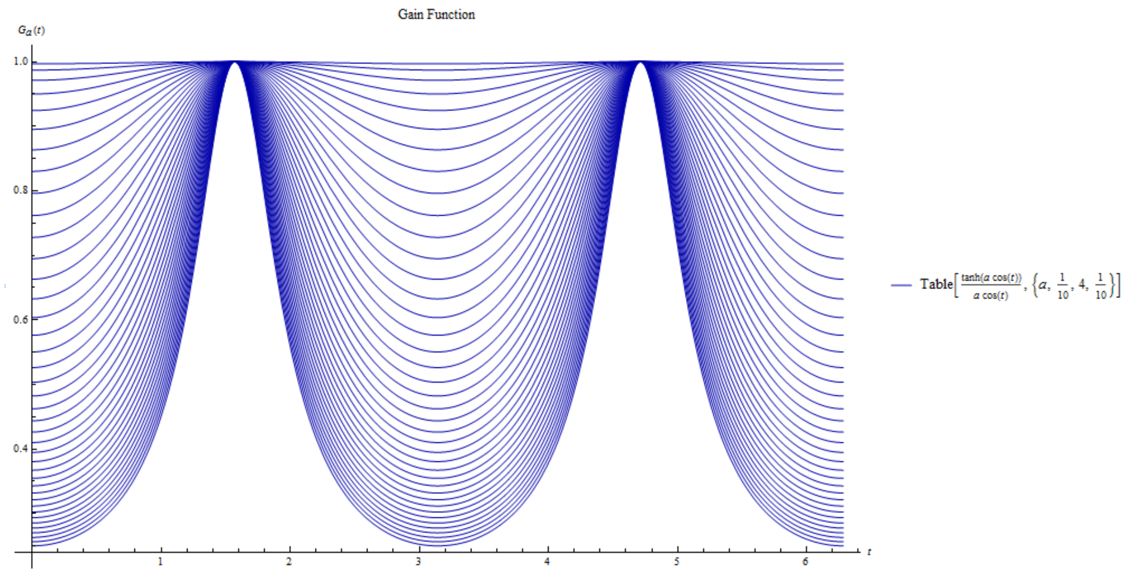
$$\Longrightarrow \cos t \,\epsilon\, \mathcal{D}(\mathcal{F}_a) \quad for\ all\ \ a \geq 0\ with\ \|\mathcal{F}_a(\cos t)\|_{L_2} \leq \sqrt{2\pi}.$$

Tanh Non-Linearity – Scattering Operator (Gain Function)

- **Reminder:**

$$\mathcal{D}(S) = \{\psi \in \mathcal{D}(\mathcal{F}) \quad | \quad \exists \mathcal{A} \in \mathfrak{P} \quad \text{with} \quad \mathcal{F}(\psi) = \mathcal{A}\psi\}$$

$$S(\psi) = A\psi.$$



$$\text{Set } (A\cos)(\text{t}) = G_a(\text{t}) \quad \textit{for all } a \geq 0$$

$$\Longrightarrow \cos t \;\epsilon\; D(S_a) \quad \textit{for all } a \geq 0 \; \textit{with} \; \|S_a(\cos t)\|_{L_2} \leq \sqrt{2\pi} \; \textit{and} \; S_a(\cos t) = G_a(\text{t}).$$

Tanh as Memoryless Non-Linear L2 System

- **Recall:**

  These are Systems that are obtained as L2 limits of the composition of a polynomial sequence defined by a fixed coefficent sequence with the input sequence.

For $c \in l_2(\mathbb{C})$ we define a non-linear operator $\mathcal{F}_c$ by

$$\mathcal{F}_c(s) = \sum_{n=0}^{\infty} c_n s^n$$

with domain of definition

$$D(\mathcal{F}_c) = \left\{ s \in L_2(0, 2\pi) \mid \lim_{N \to \infty} \sum_{n=0}^{N} c_n s^n \text{ exists as } L_2 - \text{limit} \right\}.$$

$$c_n(a) = \begin{cases} 0, & if \quad n \quad \text{is even} \\ \frac{2^{n+1}(2^{n+1}-1)B_{n+1}}{(n+1)!} a^n, & if \quad n \quad \text{is odd,} \end{cases}$$

where B_n are the Bernoulli numbers and a is the amplitude.

Tanh as Memoryless Non-Linear L2 System (Convolution Test)

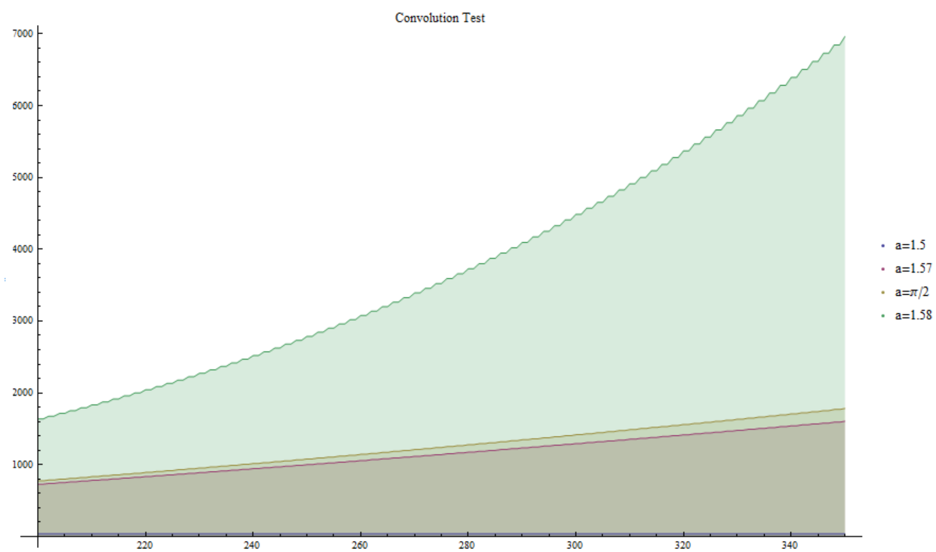
Apply:

$b_n \geq 0$ *and define* $h_n = 4^{-n}\binom{2n}{n}$ *for all* $n \in \mathbb{N}_0$.

$\mathcal{F}_b^o(\cos t) = \sum_{n=0}^{\infty} b_n \cos^{2n+1} t \in L_2(0, 2\pi)$ *iff* $\langle Lh, b * b \rangle < \infty$ *with*

$$\|\mathcal{F}_b^o(\cos t)\|_{L_2}^2 = 2\pi \, \langle Lh, b * b \rangle.$$

In addition if $\langle Lh, b * b \rangle < \infty$ *the series* $\mathcal{F}_b^e(\cos t)$ *converges a.e being also the* L_2 *limit.*



$$\Longrightarrow \cos t \in \mathcal{D}\left(\mathcal{F}_{c(a)}\right) \quad for\ all \ \ 0 \leq a \leq \pi/2 .$$

Scattering sequences

Theorem 3.2. *Under appropriate conditions on $c \in l_2(\mathbb{C})$ and if $c_0 = 0$ we obtain for $a > 0$ and $\theta \in [0, 2\pi)$*

$$m_{\mathcal{F}_c}(a\cos(t+\theta))(l) = e^{il\theta} \sum_{j=0}^{\infty} c_{2j+|l|+1} \left(\frac{a}{2}\right)^{2j+|l|} \binom{2j+|l|}{j}.$$

More general we can look at the case $\mathcal{F}_c(a\cos t) \in L_2(0, 2\pi)$ for some general $\mathcal{F}_c$. Consequently $\mathcal{F}_c(a\cos t) = \sum_{n\in\mathbb{Z}} a_n e^{int}$ with $(a_n)_{n\in\mathbb{Z}} \in l_2(\mathbb{C})$. Then formally

$$m_{\mathcal{F}_c}(a\cos(t))(l) = \frac{1}{a} \sum_{n\in\mathbb{Z}} a_n \sin\left(|n-l|\frac{\pi}{2}\right)$$

A matlab program was written for calculating the sequence with both represenations and checking their validity.

Fourier Series of Tanh(a Cos(t))

- **By elementary transformations:**

$$\frac{1}{2\pi}\int_0^{2\pi} \tanh\left(a\cos t\right)\cos\left(mt\right)\ dt = \begin{cases} 0, & if \quad n \quad \text{is even} \\ \frac{2}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(mt)}{1+e^{-2a\cos t}}\ dt - \frac{(-1)^{\left(\frac{m-1}{2}\right)}}{m}\right), & if \quad n \quad \text{is odd} \end{cases}$$

- **CRVZ convergence accleration** Method for geometric series (H. Cohen, F. R. Villegas, D. Zagier, Experiment Math., Volume 9, Issue 1, 3-12) leads to consideration of with full error control

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-2a\cos t}\cos\left(mt\right)\ dt, \quad m \text{ odd}$$

- **Possibility 1:** Expansion of $e^{-2a\cos t}$ into a Bessel I series (usable for small a).
  - Problem 1: Calculate

$$\int_0^1 \frac{T_m\left(u\right)T_n\left(u\right)}{\sqrt{1-u^2}}\ du, \quad T_n \ n\text{-th Chebyshev polynomial}$$

- **Possibility 2:** Calculate a finite Laplace transform on $[0,1]$ (usable for moderate sized a).
  - Problem 2: Calculate

$$\int_0^1 \frac{T_m\left(u\right)}{\sqrt{1-u^2}}e^{-2au}\ du$$

Fourier Series of Tanh(a Cos(t))

Problem 1 is solved by using a multiplication theorem for Chebyshev polynomial.

Problem 2 is solved by using the analytic expression for the Chebyshev polynomial and properties of derivatives of BesselI and Struve L functions.

$$\int_0^1 \frac{T_m(u)}{\sqrt{1-u^2}} e^{-2au}\, du = (-1)^{m+1}\frac{\pi}{2} M_{-m}(2a) - \mathcal{L}_{m-1}\left(\frac{1}{2a}\right)$$

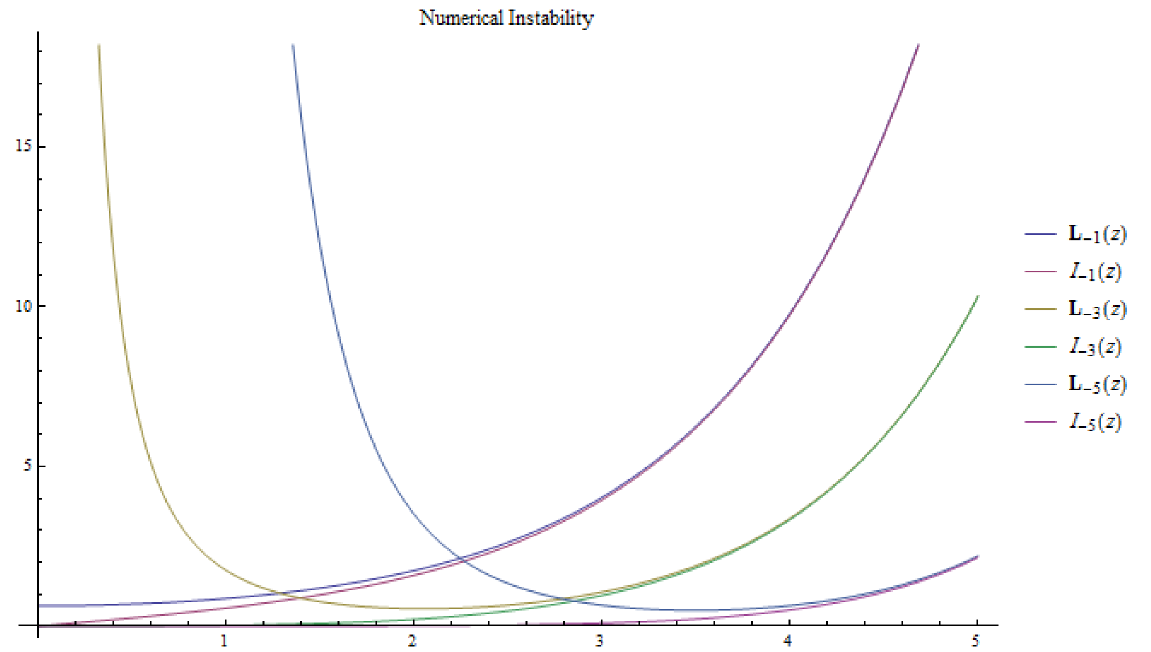
$M_m(z) = L_m(z) - I_m(z)$, $\quad L_m$ Struve L function und I_m Bessel I function of order L,

$\mathcal{L}_{m-1}$ are polynomials of degree $m-1$.

Struve L functions are not available in matlab but can be obtained by the the generalized hypergeometric function ${}_1F_2$ which is available in matlab.

Fourier Series of Tanh(a Cos(t))

Problem 3 The Definition of the **Struve M** function becomes **numerically unstable** for large arguments.



Problem 3 is solved by using the following integral representation.

$$I_{-\nu}(x)-\mathbf{L}_{\nu}(x)=\frac{2(x/2)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\infty}\sin(tx)(1+t^2)^{\nu-\frac{1}{2}}\,dt \qquad (\mathcal{R}\nu<\tfrac{1}{2}, x>0)$$

Open Problems

- Scattering sequence for $\tanh(a \cos t)$ for amplitudes $a > \frac{\pi}{2}$.
- Is were another representation for the scattering sequence than in the memoryless case and that is the relation between both.