Linear periodic differential equations and X-parameters

Hans Georg Brachtendorf

University of Applied Sciences of Upper Austria
Hardware Software Design
FH-OÖ/Hagenberg

Basic Notation · Nonlinear ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) + f\big(x(t)\big) - b(t) = 0, \quad x(t_0) = x_0,\, x = (v, i)^T$$

with periodic (large signal) stimulus

$$b(t) = b(t+T),\ \omega_0 = \frac{2\pi}{T}$$

and steady state solution

$$x_{ss}(t) = x_{ss}(t+T)$$

Remark

The nonlinear ODE system is time-invariant.

Basic Notation · perturbation

Perturbation of the stimulus

$$\hat{b}(t) = b(t) + \Delta b(t), \quad \Delta b(t) = \Delta b(t+T)$$

with the same period, *arbitrary phase*, possibly *higher harmonics*, where $\|\Delta b\| \ll \|b\|$
Linearization of the ODE assuming small signal assumption

$$\dot{x}(t) = A(t)x(t) + \Delta b(t)$$

where

$$A(t) := -f_x(x_{ss}(t)), \quad A(t) = A(t+T)$$

the Jacobian matrix of $f\left(x\right)$ (linearization) evaluated at steady state x_{ss}.

Basic Notation · Linear periodic ODEs

$$\dot{x}(t) = A(t)x(t) + \Delta b(t), \quad \Delta b(t) = \Delta b(t+T)$$

is a *linear time periodic variational* ODE system

Basic Notation · X-parameter measurement

X parameter measurements: large signal stimulus at port 1, namely b.
Small signal "ticky stimulus" at an arbitrary port, namely Δb
Variational equation is a linear and time-periodic ODE system

$$\dot{x}(t) = A(t)x(t) + \Delta b(t), \quad \Delta b(t) = \Delta b(t+T)$$

Remark

The perturbation ODE system is not time-invariant! The large signal stimulus at port 1 operates as a reference clock!

Linear time periodic differential equations

$$\dot{x}(t) = A(t)\,x(t) + \Delta b(t), \quad \Delta b(t) = \Delta b(t+T)$$

with general solution (zero-input and zero-state parts)

$$\varphi(t, x(t_0)) = \Phi(t,\, t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)\,b(\tau)\,\mathrm{d}\tau$$

where the *state transition matrix* reads

$$\Phi(t,\tau) = U(t)\,\Lambda(t-\tau)\,V(\tau),$$
$$\Lambda(t-\tau) = \mathrm{diag}\Big(\exp\big(\mu_1(t-\tau)\big), \ldots, \exp\big(\mu_n(t-\tau)\big)\Big)$$

Stability requires that all Floquet exponents $\mathrm{Re}\{\mu_i\} < 0$

Linear time periodic differential equations

$$\varphi(t, x(t_0)) = \Phi(t, t_0)\, x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\, b(\tau)\, \mathrm{d}\tau$$

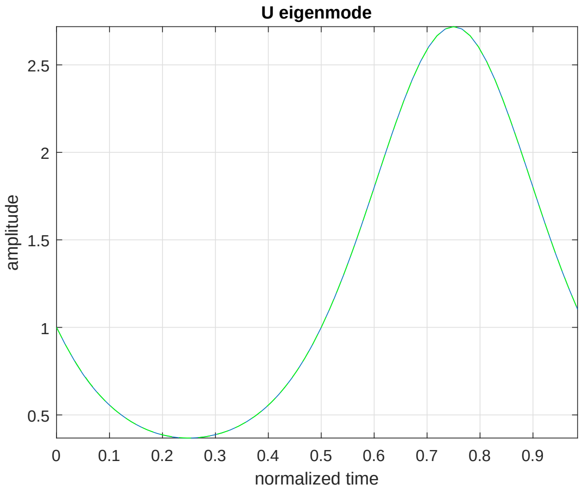
Moreover the matrices U, V are periodic and bi-orthogonal, i.e.

$$V(t) = V(t+T), \quad V(t) = U^{-1}(t)$$

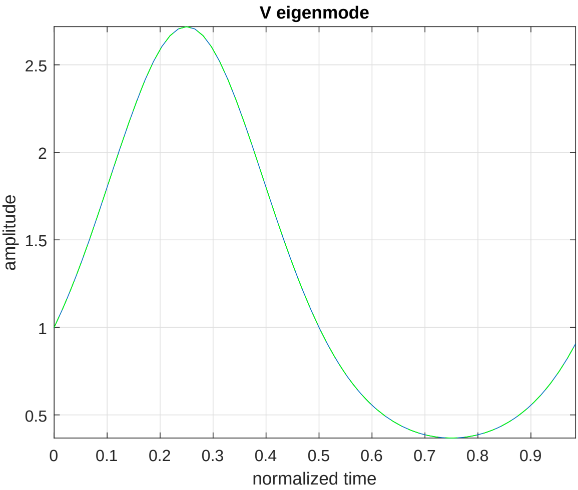
The column vectors of $U(t)$, vice versa row vectors of $V(t)$ (eigenmodes) and Floquet exponents μ_i control the dynamics of the system
Choose specific steady state solution of the variational problem where

$$\Delta x_{ss}(t) = \varphi(t \to \infty, x(t_0)) = \Delta x_{ss}(t+T)$$

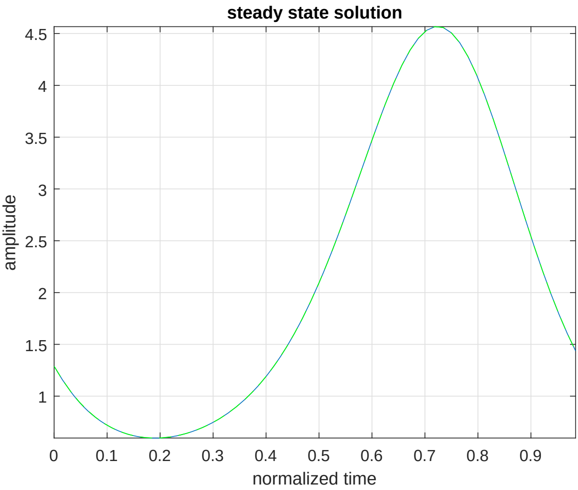
Simulation results · Numerics vs. analytics



Simulation results · Numerics vs. analytics



Simulation results · Numerics vs. analytics



Next steps

- Generalization to differential-algebraic equations (DAEs)
- Choice of the significant eigenmodes of the system employing MOR
- Derivation of the Floquet exponents and eigenmodes from measurements