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Non Linear Scattering Matrices

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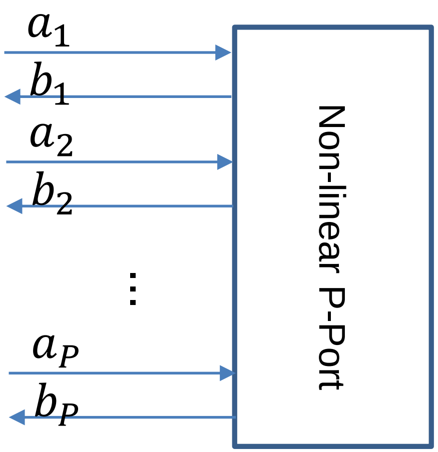
Definition of Non-Linear Scattering Matrices

Scattering Variables a, b (see [1])



$$a_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{Hi} \end{pmatrix},\ b_i = \begin{pmatrix} b_{1i} \\ \vdots \\ b_{Hi} \end{pmatrix} \in \mathbb{C}^H,\ i = 1, \dots, P,\ H\ number\ of\ Harmonics$$

Matrices A, B of in and outgoing scattering states

$$A = (a_1\ \cdots\ a_P),\ \text{B}= (b_1\ \cdots\ b_P) \in \mathbb{C}^{H\times P},\ a = vec(A), b = vec(B) \in \mathbb{C}^{HP}$$

In case of infinite harmonics the vec operation makes no sense and one has to redefine the matrices A and B as the transpose. The above definition corresponds to the formulation in [1].

Definition of Non-Linear Scattering Matrices

Non linear scattering matrix (see [1],[2])

$$b(A) = S(\mathrm{A})a, S(A) \in \mathbb{C}^{HP\times HP}$$

- Existence of $S(\mathrm{A})$ follows from global scattering solutions satisfying asymptotic boundary conditions (eg. for nonlinear interactions in quantum scattering theory see [2]).
- If S does not depend on A this equation determines S uniquely.
- For nonlinear interactions S depends on A and uniqueness cannot be ensured.
- Minimal assumption : $S(\mathrm{A})$ is defined and continuous in A on some domain $\Omega \ni 0$.

Example

- Two port network with 3 harmonics

$$b(A) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = S(A)a$$

$$S_{ij} = \begin{pmatrix} S_{ij11} & S_{ij12} & S_{ij13} \\ S_{ij21} & S_{ij22} & S_{ij23} \\ S_{ij31} & S_{ij32} & S_{ij33} \end{pmatrix}$$

same frequency scattering parameter

frequency down conversion

port index of b vector

port index of a vector

harmonic index of b vector

harmonic index of a vector

frequency up conversion

Inverse Scattering Problem for 2 Ports and One Frequency

Problem: Is it possible to determine $S(\mathrm{A})$ from the scattering variables $b(A)$?

Let‘s have a look at the two port case with one frequency index.

Here $\mathrm{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and

known scattering data

$$\mathrm{b(A)} = \mathrm{b}(a_1, a_2) = \begin{pmatrix} b_1(a_1, a_2) \\ b_2(a_1, a_2) \end{pmatrix} = \begin{pmatrix} S_{11}(a_1, a_2) & S_{12}(a_1, a_2) \\ S_{21}(a_1, a_2) & S_{22}(a_1, a_2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

unknown elements of the scattering matrix

and S is assumed to be continuous on $\Omega_1 \times \Omega_2 \ni 0$.

For $(a_1, a_2) \in \Omega_1 \setminus\{0\} \times \Omega_1\setminus\{0\} set$

$$S_0\ (a_1, a_2) = \left(\frac{b\ (a_1, 0)}{a_1} \quad \frac{b\ (0, a_2)}{a_2} \right).$$

Then S_0 extends continuously to $\Omega_1 \times \Omega_2$ with

$$\lim_{a_1 \to 0} \frac{b\ (a_1, 0)}{a_1} = \frac{\partial b}{\partial a_1}(0,0) = \begin{pmatrix} S_{11}(0,0) \\ S_{21}(0,0) \end{pmatrix} \text{ and } \lim_{a_2 \to 0} \frac{b\ (0, a_2)}{a_2} = \frac{\partial b}{\partial a_2}(0,0) = \begin{pmatrix} S_{21}(0,0) \\ S_{22}(0,0) \end{pmatrix}$$

Inverse Scattering Problem for 2 Ports and One Frequency, S_0 Matrix

By definition

$$S_0\ (0{,}0) = S\,(0{,}0) \text{ and for } (a_1, a_2) \in \Omega_1 \times \Omega_2 \subset \mathbb{C}^2$$

$$S_0\ (a_1, 0)\begin{pmatrix} a_1 \\ 0 \end{pmatrix} = S\,(a_1, 0)\begin{pmatrix} a_1 \\ 0 \end{pmatrix} \text{ resp. } S_0\ (0, a_2)\begin{pmatrix} 0 \\ a_2 \end{pmatrix} = S_0\ (0, a_2)\begin{pmatrix} 0 \\ a_2 \end{pmatrix}.\ (*)$$

Such an S_0 is not uniquely defined. For continuous functions Φ_{11}, Φ_{12}, Φ_{21}, Φ_{22} on with $\Omega_1 \times \Omega_2$ $\Phi_{11}(a_1, 0)$=$\Phi_{21}(a_1, 0)$=0 resp. $\Phi_{12}\ (0, a_2)$=$\Phi_{22}\ (0, a_2)$=0 define

$$\delta S_0\ (a_1, a_2) = \begin{pmatrix} \Phi_{11}(a_1, a_2) & \Phi_{12}(a_1, a_2) \\ \Phi_{21}(a_1, a_2) & \Phi_{22}(a_1, a_2) \end{pmatrix}$$

Then $S_0 + \delta S_0$ also satisfies $(*)$. Such δS_0 is obtained by $\delta S_0\ (a_1, a_2) = S(a_1, a_2)$-$S_0(a_1, a_2)$ which trivially satisfies $S_0(a_1, a_2) + \delta S_0\ (a_1, a_2) = S(a_1, a_2)$ but since S is not known this is of no help.

Inverse Scattering Problem for 2 Ports and One Frequency, S_0 Matrix

Summary

- S_0 depends only on scattering data.
- S_0 is continuous on $\Omega_1 \times \Omega_2$.
- S_0 coincides with the scattering matrix in the linear case and is uniquely defined.
- S_0 is non unique in the non-linear case.

Inverse Scattering Problem for 2 Ports and One Frequency, ΔS Matrix

Problem: S_0 does not satisfy

$$S_0\ (a_1, a_2)a = S(a_1, a_2)a.$$

Goal: Find such a matrix with this property which depends only on scattering data

Idea: Since S_0 coincides with S in the linear case we split off S_0 from S and define

$$\Delta S(a_1, a_2) = S(a_1, a_2) - S_0\ (a_1, a_2)$$

Then

$\Delta S(a_1, a_2)a = \Delta b(a_1, a_2)$ with $\Delta b(a_1, a_2) = b(a_1, a_2)$-$b(a_1, 0)$-$b(0, a_2)$.

Try to solve for the matrix components of $\Delta S(a_1, a_2)$!

Inverse Scattering Problem for 2 Ports and One Frequency, ΔS Matrix

Solution: For $all\ (a_1, a_2) \in \Omega_1 \setminus\{0\} \times \Omega_1\setminus\{0\}$

$$\Delta S(a_1, a_2) = D(a_1, a_2) + R(a_1, a_2)$$

where $D(a_1, a_2) = \mathrm{diag(a)}^{-1}\mathrm{diag}(\Delta b(a_1, a_2))$ is diagonal and

$$R(a_1, a_2) = \begin{pmatrix} \Phi_1(a_1, a_2) \\ -\Phi_2(a_1, a_2) \end{pmatrix} \begin{pmatrix} -a_2 & a_1 \end{pmatrix} \quad (**)$$

is a rank one matrix with

$$\Phi_1(a_1, a_2) = \frac{\Delta S_{12}(a_1,a_2)}{a_1} \text{ and } \Phi_2(a_1, a_2) = \frac{\Delta S_{21}(a_1,a_2)}{a_2}.$$

Observe that $a \in Ker(R(a_1, a_2))$ implying $\Delta S(a_1, a_2)a = D(a_1, a_2)a$. As before $\Delta S(a_1, a_2)$ is not uniquely defined in the non-linear case since a rank one matrix of the form $(**)$ with arbitrary functions $\Phi_1(a_1, a_2)$, $\Phi_2(a_1, a_2)$ can be added.

Inverse Scattering Problem for 2 Ports and One Frequency, Final Solution

If $a_1 = 0$ or $a_2 = 0$ set

$$S(a_1, a_2) = S_0(a_1, a_2),$$

otherwise set

$$SD(a_1, a_2) = S_0(a_1, a_2) + D(a_1, a_2).$$

Then

$$SD(a_1, a_2)a = S(a_1, a_2)a \text{ holds for all } (a_1, a_2) \in \Omega_1 \times \Omega_2.$$

Inverse Scattering Problem for 2 Ports and One Frequency, Final Solution

Summary

- $SD(a_1, a_2)$ depends only on scattering data.
- $SD(a_1, a_2)$ coincides with the scattering matrix in the linear case since then $\Delta b(a_1, a_2) = 0$ and is uniquely defined.
- $SD(a_1, a_2)$ is non unique in the non-linear case.

To do: Is $SD(a_1, a_2)$ continuous at zero or more general how can it be made continuous at zero?

To Do List

Show continuity or find a continuous representative

Try to extend the above reasoning to more ports and more frequencies

Connection to X Parameters?

Connection to Volterra Series?

Connection to Floquet theory?

Connection to IMD and Interception Points?

Literature

- [1] Jeffrey A. Jargon, Donald C. DeGroot, K. C. Gupta, Frequency-Domain Models for Nonlinear Microwave Devices Based on Large-Signal Measurements, Journal of Research of the National Institute of Standards and Technology, Volume 109, Number 4, July-August 2004
- [2] Ali Mostafazadeh, Nonlinear Scattering and Its Transfer Matrix Formulation in One Dimension, Eur. Phys. J. Plus (2019) 134: 16. https://doi.org/10.1140/epjp/i2019-12456-x